

AUTOMORPHISMS AND EXAMPLES OF COMPACT NON-KÄHLER MANIFOLDS

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ABSTRACT. If f is an automorphism of a compact simply connected Kähler manifold with trivial canonical bundle that fixes a Kähler class, then the order of f is finite. We apply this well known result to construct compact non Kähler manifolds. These manifolds contradict the abundance and Iitaka conjectures for complex manifolds.

INTRODUCTION

Let X be a compact complex manifold of dimension n . The generalized version of the abundance conjecture says that if X is Kähler then the numerical dimension of the canonical bundle K_X should be equal to its Kodaira dimension [Dem09, Chapter 18]. A consequence of this conjecture is the Iitaka $C_{n,m}$ conjecture, which says that if $f : X \rightarrow Y$ is a holomorphic morphism of compact Kähler manifolds, then $\kappa(X) \geq \kappa(Y) + \kappa(f_y)$, where f_y is a general fiber of f and κ denotes the Kodaira dimension.

These conjectures were originally stated for projective varieties, but their statements make sense for Kähler manifolds and indeed any compact complex manifold. In this paper we produce a examples of compact non Kähler manifolds that violate both the abundance and the Iitaka conjectures. That these conjectures fail for non-Kähler manifolds has been known for some time; [Uen75, Remark 15.3] contains an example of a torus bundle over a torus that contradicts the Iitaka conjecture (it makes no mention of abundance, simply because it hadn't been conjectured at the time). The construction of our manifolds is in the same spirit as this previous example.

The construction is simple. A folklore result says that if M is a simply connected Kähler manifold with trivial canonical bundle that admits an automorphism f of infinite order, then f must move every Kähler class on M . Given such a manifold, we let a lattice in a complex vector space V act on $M \times V$ by translation on V and by mapping each generator of the lattice to f . The quotient manifold is then a compact non Kähler manifold, with flat canonical bundle, but whose Kodaira dimension is negative in some cases.

We start by detailing this construction and proving our claims on the canonical bundle, then we point to beautiful work of Oguiso that shows that the required Kähler manifolds and automorphisms exist.

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1. AUTOMORPHISMS AND KÄHLER CLASSES

Let M be a compact simply connected Kähler manifold of complex dimension $\dim_{\mathbb{C}} M = n$ with trivial canonical bundle. Examples of such manifolds include K3 surfaces, Calabi–Yau manifolds and hyperkähler manifolds; see [Bea83].

Let ω be a Kähler metric on M . The Ricci curvature of M may be defined as the curvature form of the metric that ω induces on the canonical bundle of M . In local coordinates, one has $2\pi \operatorname{Ric} \omega = -i\partial\bar{\partial} \log \det \omega_{j\bar{k}}$. Yau proved in [Yau78] that if $[\omega]$ is a Kähler class on M , then there exists a unique Ricci-flat Kähler metric ω in the class $[\omega]$. The existence of such metrics has great consequences for the geometry of the manifold M , for example:

Proposition 1.1. *An automorphism f of M fixes a Kähler class $[\omega]$ on M if and only if the order of f is finite.*

Proof: The condition is clearly sufficient, since if the degree of f is d then the Kähler class $[\omega] + f^*[\omega] + \cdots + (f^*)^{d-1}[\omega]$ is invariant under f .

Suppose now that f fixes a Kähler class $[\omega]$ and let ω be the unique Ricci flat metric in this class. Then $f^*\omega$ is again Ricci flat, and thus equal to ω by unicity. Thus f is an element of the isometry group of (M, ω) . A general result of Riemannian geometry [Bal06, Corollary 6.2] now says that the isometry group of a simply connected manifold with non positive Ricci curvature is finite. \square

The condition that M be simply connected serves to exclude complex tori, for tori admit nonzero holomorphic vector fields. These fields generate automorphisms homotopic to the identity, which thus act trivially on the cohomology of the torus, despite usually being of infinite order.

This result points the way to a construction of non-Kähler manifolds: Let M be a compact simply connected Kähler manifold with trivial canonical bundle. Suppose M admits an automorphism f of infinite order. Let V be a complex vector space of dimension p and let Γ be a lattice in V , we denote by $B = V/\Gamma$ the complex torus defined by Γ . We define a representation $\Gamma \rightarrow \operatorname{Aut} M$ by mapping every generator of Γ to the automorphism f . The lattice Γ then acts on the product $M \times V$ by

$$\gamma \cdot (z, t) = (\gamma(z), t + \gamma).$$

We set $X := X(M, B) = (M \times V)/\Gamma$.

Proposition 1.2. *The complex space X is a smooth compact non Kähler manifold. It is the total space of a holomorphic fibration $\pi : X \rightarrow B$, whose fibers are all isomorphic to M .*

Proof: The lattice Γ clearly acts without fixed points on $M \times V$. Its action is also properly discontinuous, since any compact set in $M \times V$ may be translated as far to infinity in V as desired. The quotient X is thus a smooth complex manifold, and compact for the same reason that the torus V/Γ is compact.

The projection map $pr : M \times V \rightarrow V$ is invariant by the action of Γ and thus defines a holomorphic morphism $\pi : X \rightarrow B$. It is proper as the manifold X is compact, and a submersion because the projection morphism is a submersion. Let t be a point of B . The preimage $\pi^{-1}(t)$ may be

identified with the product $M \times \Gamma + t$. If we pick an element γ in the lattice Γ , then the restriction of the quotient map $q : M \times V \rightarrow X$ identifies with the automorphism $\gamma \cdot f : M \rightarrow M$ and defines an isomorphism $M \rightarrow X_t$.

Finally, suppose that X were Kähler. If ω were a Kähler metric on X , then by restriction we would obtain a Kähler class $[\omega_0]$ on the fiber M_0 that would be invariant under the action of the monodromy on the cohomology of M_0 . But the monodromy group is the same as the group generated by f , so this is impossible since f is of infinite order. \square

Remark — It seems hard to extract precise topological information about X , aside from that which follows trivially from general facts about fibrations. For example, the naive road to the Betti numbers of X passes through the space of closed forms on $M \times V$ that are invariant under the automorphism f . Since f is quite wild I have no idea how one could calculate this in practice.

The canonical bundle of M is trivial, so there is a nowhere zero holomorphic $(n, 0)$ -form σ on M . As $f^*\sigma$ is again a $(n, 0)$ -form on M , we must have $f^*\sigma = \lambda\sigma$ for some complex number λ . Note that the (n, n) -form $i^{n^2}\sigma \wedge \bar{\sigma}$ is real and positive on M , and that $f^*(\sigma \wedge \bar{\sigma}) = |\lambda|^2\sigma \wedge \bar{\sigma}$. Integrating over M , we find $|\lambda| = 1$.

Proposition 1.3. *The Kodaira dimension of X is zero if λ is a root of unity and negative otherwise.*

Proof: Suppose α is a global section of mK_X for some $m \geq 1$. If $q : M \times V \rightarrow X$ is the quotient map, then $q^*\alpha$ is a global section of $mK_{M \times V}$. We may thus write

$$q^*\alpha = \theta(z, v) (\sigma_M \otimes \sigma_V)^{\otimes m},$$

where σ_M is a trivializing section of K_M , $\sigma_V = dv_1 \wedge \dots \wedge dv_n$ is the standard holomorphic volume form on V , and θ is a holomorphic function on $M \times V$. We note that since M is compact, θ is actually just a holomorphic function on V .

Since α is a section of mK_X , the pullback $q^*\alpha$ must be invariant under the action of Γ on $M \times V$. The holomorphic volume form σ_V is invariant under the action of Γ , so if γ_i is one of the generators of Γ we find

$$\theta(v) (\sigma_M \otimes \sigma_V)^{\otimes m} = q^*\alpha = \gamma_i \cdot q^*\alpha = \lambda^m \theta(v + \gamma_i) (\sigma_M \otimes \sigma_V)^{\otimes m}.$$

If $\gamma = \sum_i a_i \gamma_i$ is an element of Γ , we set $\deg \gamma := \sum_i a_i$. Using the above we then get $\theta(v) = \lambda^{m \deg \gamma} \theta(v + \gamma)$ for any γ and v . This entails that $|\theta(v)| = |\theta(v + \gamma)|$ for all v and γ , but then $|\theta|$ takes its maximum on V in the fundamental parallelogram of Γ , so θ is constant. The complex number λ must then satisfy $\lambda^m = 1$.

We thus see that if λ is an m^{th} root of unity, then every m^{th} power of K_F admits a unique non-zero holomorphic section, so the Kodaira dimension of X is zero. Likewise, if λ is not a root of unity, then no power of K_M admits a global section, so the Kodaira dimension of X is negative. \square

Proposition 1.4. *The numerical dimension of K_X is zero.*

Proof: We will show that the canonical bundle K_X admits a flat hermitian metric. Its first Chern class is thus zero, which implies the proposition.

Since $M \rightarrow X \rightarrow B$ is a fibration there is a short exact sequence

$$0 \longrightarrow T_{X/M} \longrightarrow T_X \longrightarrow \pi^* T_B \longrightarrow 0$$

of tangent bundles over X . Note that since B is a torus the bundle $\pi^* T_B$ is trivial. The adjunction formula now says that the canonical bundle of X is $K_X = K_{X/M}$. Let $q : M \times V \rightarrow X$ be the quotient morphism and consider the pullback bundle $q^* K_{X/M} = p_M^* K_M$, where $p_M : M \times V \rightarrow M$ is the projection.

Now pick a Ricci-flat Kähler metric ω on M , and let $dV = \omega^n/n!$ be its volume form. Recall that the volume form of any other Ricci-flat Kähler metric is a constant multiple of dV . The form dV defines a smooth hermitian metric on $p_M^* K_M$ by the formula $h(\alpha, \bar{\beta}) dV = i^{n^2} \alpha \wedge \bar{\beta}$, where α and β are sections of $p_M^* K_M$. The curvature form of this metric is the Ricci-form of ω , so it is flat.

If σ_M is a trivializing holomorphic volume form on M , then $f^* \sigma_M = \lambda \sigma_M$, where λ is a complex number with absolute value 1. Also note that $f^* \omega$ is again a Ricci-flat Kähler metric on M , and that

$$\text{Vol}(M, f^* \omega) = \int_M \frac{f^* \omega^n}{n!} = \int_M \frac{\omega^n}{n!} = \text{Vol}(M, \omega)$$

because $f : M \rightarrow M$ is a surjective finite morphism of degree one. Thus $f^* dV = dV$. From these two facts it follows that

$$f^*(h(\alpha, \bar{\beta})) dV = f^*(h(\alpha, \bar{\beta}) dV) = i^{n^2} f^* \alpha \wedge \overline{f^* \beta} = h(f^* \alpha, \overline{f^* \beta}) dV,$$

so the metric h is invariant under the action of Γ and thus defines a flat hermitian metric on $K_{X/M} = K_X$. \square

2. AUTOMORPHISMS OF HYPERKÄHLER MANIFOLDS

As before we let M be a compact simply connected Kähler manifold with trivial canonical bundle. The automorphism group of M admits a natural representation

$$\text{Aut } M \longrightarrow \text{Aut } H^2(M, \mathbb{C}),$$

obtained by sending each automorphism to the pullback morphism on cohomology. If M is a K3 surface, then the global Torelli theorem entails that this group morphism is actually injective. The order of an automorphism f is thus equal to the order of its pullback f^* on degree two cohomology.

One may obtain examples of higher dimensional holomorphic symplectic manifolds from a K3 surface, see [Bea83]. The idea is to consider the symmetric product M^n/\mathfrak{S}_n . This space is singular, but the Douady space $M^{[n]}$ of subspaces of M of length n is a desingularization of the symmetric product. The Douady space is then a holomorphic symplectic manifold of dimension $2n$.

The second cohomology of the Douady space is isomorphic to

$$H^2(M^{[n]}, \mathbb{C}) = H^2(M, \mathbb{C}) \oplus \mathbb{C} \cdot E,$$

where E is an exceptional divisor of the desingularization $M^{[n]} \rightarrow M^n/\mathfrak{S}_n$. Any automorphism f of the K3 surface M induces an automorphism of the Douady space $M^{[n]}$. This new automorphism acts like f on the part of the second cohomology coming from M , and trivially on the exceptional divisor.

In particular, if f is of infinite order on M , then the induced automorphism on $M^{[n]}$ is of infinite order.

Recall that the holomorphic symplectic form σ on M is unique up to scalars. It follows that σ is an eigenvector of any automorphism f of M , and as before one sees that the eigenvalue of σ must have absolute value 1. Oguiso gives much more precise results in [Ogu08]; for the moment we will contend ourselves with the following special case of his Theorem 2.4:

Proposition 2.1. *Let M be a projective K3 surface and f an automorphism of M . Let λ be the eigenvalue of f^* on the space $H^0(M, K_M)$. Then λ is a root of unity.*

By our discussion of Douady spaces, the same is true of the holomorphic symplectic space constructed from a projective K3 surface.

Example 1. Let $\mathbb{P} := \mathbb{P}_1^1 \times \mathbb{P}_2^1 \times \mathbb{P}_3^1$. This space comes equipped with three projections $p_i : \mathbb{P} \rightarrow \mathbb{P}_i$. Let

$$L := p_1^* \mathcal{O}(1)_{\mathbb{P}_1^1} \otimes p_2^* \mathcal{O}(1)_{\mathbb{P}_2^1} \otimes p_3^* \mathcal{O}(1)_{\mathbb{P}_3^1}$$

be an ample line bundle on \mathbb{P} , so that $K_{\mathbb{P}} = -2L$. The adjunction formula shows that if τ is a general section of $2L$, then the zero variety $X = \tau^{-1}(0)$ is a smooth K3 surface.

We can now consider the projections $p_{jk} : \mathbb{P} \rightarrow \mathbb{P}_j \times \mathbb{P}_k$. Restricted to the K3 surface M , these define ramified coverings $M \rightarrow \mathbb{P}_j \times \mathbb{P}_k$ of degree 2. The Galois groups of these coverings give three holomorphic involutions ι_i of M , and we have

$$\text{Aut } X = \langle \iota_1, \iota_2, \iota_3 \rangle \simeq \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2,$$

where $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. Both identities in the above formula are nontrivial, but they are proved in [Ogu11a]. The automorphism group of M thus contains several elements of infinite order.

Example 2. We again refer to Oguiso's paper [Ogu08, Examples 2.5 and 2.6], from which one may extract that there exists a K3 surface M which admits an automorphism f such that the eigenvalue of f^* on $H^0(M, K_M)$ has infinite order. As before, it follows that there exist higher dimensional hyperkähler manifolds with the same property.

We now consider the non Kähler manifold $X = X(M, B)$. This manifold has negative Kodaira dimension by our earlier results. By construction there is a holomorphic map $\pi : X \rightarrow B$ whose fiber at every point is M . Both M and B have Kodaira dimension zero, so

$$\kappa(X) < \kappa(M) + \kappa(B).$$

The manifold X thus shows that the Iitaka $C_{n,m}$ conjecture is false for general compact complex manifolds. Since $\kappa(X)$ is negative but the canonical bundle K_X has numerical dimension zero, the manifold also shows that the generalized abundance conjecture is false for general complex manifolds.

Example 3. Oguiso and Schröer show in [Ogu11b] that the universal cover of the Douady space $M^{[n]}$ of an Enriques surface M is a Calabi–Yau manifold. They also show that there exists an Enriques surface M with $\text{Aut } M = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, similarly to the hyperkähler manifolds considered

above. The fibration $X(M^{[n]}, B)$ then provides an example of a non-Kähler manifold $X \rightarrow B$ with a Calabi–Yau fiber.

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